Passive cyclotron current drive efficiency for relativistic toroidal plasmas

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In this paper we present results of analytical and numerical studies of the passive cyclotron current drive efficiency in mildly relativistic toroidal plasmas. The problem of linearization and separation of the electron and photon balance equations becomes nontrivial for high-temperature plasmas (e.g., D–^{3}He) with low electron pressure ($\beta_e<0.1$) due to the increased effect of radiation friction. The conditions under which this separation is possible is derived in this paper. The linearized problem for the electron distribution is formulated in the form of a standard variational principle, which includes both Coulomb collisions and ''collisions'' due to cyclotron radiation. The reduced variational principle for the current drive efficiency (generalized Spitzer–Härm function) is derived, as well as its bounce-averaged form for toroidal plasmas. Finally, a convenient form of the passive cyclotron current drive efficiency is introduced, which can be used for a self-consistent modeling of passive cyclotron current generation in tokamaks with the help of fish-scale structures [see the companion paper, W. Kernbichler and S. Kasilov, Phys. Plasmas **3**, 4128 (1996)]. [S1070-664X(96)04311-X]

I. INTRODUCTION

The passive cyclotron current drive method proposed for toroidal plasma traps[1] has been extensively studied in the literature.[2–5] These studies show that the amount of driven current can be sufficient for a tokamak-reactor based on D–^{3}He fusion schemes with high operational electron temperature $T_e>40$ keV, staying relatively small for D–T schemes, with T_e in the order of 10 keV. But the method also looks attractive as a source of a seed current, especially for tight aspect ratio tokamaks with a large amount of bootstrap current, because the main amount of current is driven in the center of the plasma. This was shown in Ref. 3 with the help of scaling laws, and it is shown in Refs. 6 and 7 in more detailed analyses.

The current sustainment method proposed in Ref. 1 assumes a ''fishscale'' structure of the reflecting wall of the plasma chamber, which provides an asymmetric wall reflectivity for the cyclotron radiation in a toroidal direction, thus creating the nonzero total parallel momentum of the photon population. Finally, as a result of the interaction with the plasma electrons a current is driven in the plasma. As it was originally seen in the literature, the presence of fish-scale structures should lead to an increase of cyclotron energy losses from the plasma due to the decrease of the effective wall reflectivity. The optimum current production should be expected at small fish-scale angles in the order of $\Theta_F=(1-\Gamma)/2$, where Γ is the reflectivity of the reflecting surface.[1]

The more detailed studies of the local efficiency of the passive cyclotron current drive method[2] showed that the highest efficiency is obtained if the reflected radiation propagates at an angle $\theta=40°$ with respect to the magnetic field. The particular way of creation of such a narrow radiation distribution was not discussed in this study, but it looks impossible with a simple wall structure, as proposed in Ref. 1. The angular distribution of radiation created by fish-scale structures is rather broad over θ and varies with the position in the plasma. Thus, in order to obtain reliable values of the plasma current and its profile, one has to take into account the wall structure in a consistent way. Such a study has been performed numerically[6,7] for an inhomogeneous cylindrical plasma column surrounded by a reflecting wall with a fish-scale structure. The detailed derivation of the passive cyclotron current drive efficiency used in that study is given in the present paper.

The previous studies of the local efficiency of passive cyclotron current drive were performed in Ref. 3, with the assumption that current is produced mainly by suprathermal electrons, which allows for the use of the asymptotical formula for the collision frequency.[8] In particular, these studies showed that besides the direct momentum transfer from the radiation to suprathermal electrons, an important role is played by the Fisch–Boozer effect,[9] which contributes an additional amount of current. In addition, the effect of toroidicity, which should decrease the plasma current, was shown to be not dramatic[4] for the current generated by radiation propagating at $\theta=40°$, which was recommended in Ref. 3.

In the present paper the computation of the efficiency was performed without the asymptotical approximation, and therefore, in particular, includes the response of the background electrons, which partly recover the parallel momentum lost by the tail electrons, thus increasing the current drive efficiency. The role of this effect is more important for high-temperature plasmas because of the relativistic saturation of the current drive efficiency for suprathermal electrons.

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Another important effect that was taken into account is the local relaxation of electrons by means of cyclotron radiation. Usually this effect is neglected in the literature on cyclotron losses and current drive. The argument for that usually is that the radiation friction force F_r is small enough compared to the Coulomb one F_C. So this does not significantly distort the electron distribution from the Maxwellian one. This basic assumption allows us to linearize the electron–photon equilibrium problem, considering separately the distribution of radiation intensity in the Maxwellian plasma and then finding current from the linearized kinetic equation, which includes the effect of radiation in the form of a quasilinear source term (see, e.g, Ref. 3).

The small parameter of this approximation $F_r/F_C \sim (4/\Lambda\beta_e)(T_e/mc^2)(\gamma^2 v^3/c^3)$ may be violated for high-temperature plasmas with a small electron pressure $\beta_e = 8\pi n_e T_e/B^2$ in the order of a few percent, because the tail electrons mainly responsible for the losses and the generation of current are already sufficiently relativistic $(r \sim c)$. However, as is shown below, the linearization stays valid, even in this case, if the optical thickness of the plasma is big enough in the low-frequency region. This permits us to consider the interaction between the electrons through cyclotron radiation as a sufficiently local process, described with the help of a radiation collision integral, derived in the nonlinear case in Refs. 10–12. The nonlocal radiation effects that distort the Maxwellian distribution due to the finite plasma size stay small. The small parameter of this approximation is derived in Sec. III. In this section, the general nonlinear problem of the electron–photon equilibrium is also linearized.

As a result, the kinetic equation contains the linearized radiation collision operator (derived in Sec. IV), together with the quasilinear source term and the Coulomb collision operator. Instead of this kinetic equation, an equivalent variational principle was introduced, similar to the one used in Refs. 13–15.

The reduced variational principle for the current drive efficiency (generalized Spitzer–Härm function) is derived in Sec. V. In the same section the passive cyclotron current drive efficiency, which is related to the radiation intensity, is introduced. The bounce-averaged problem for the generalized Spitzer–Härm function is formulated in Sec. V. Finally, the numerical method and the results are presented and discussed in Sec. VII.

II. FORMULATION OF THE PROBLEM

For simplicity, the coupled problem of the electron–photon equilibrium is considered here for the case of a uniform magnetic field and a uniform plasma along the magnetic field lines. In this case the electron distribution function f obeys the stationary kinetic equation,

$$\frac{\partial f}{\partial t} = \hat{L}_C(f,f) + \hat{L}_{cy}(I,f) = 0, \tag{1}$$

with the relativistic Coulomb collision integral $\hat{L}_C$ and the electromagnetic interaction operator for cyclotron radiation $\hat{L}_{cy}$. The photon distribution is governed by the equation for the radiation intensity, which, neglecting refraction, can be written as

$$\mathrm{div}[\mathbf{k} I_\omega^{(M)}(\theta,\phi)] = j_\omega^{(M)}(\theta,\phi) - \alpha_\omega^{(M)}(\theta,\phi) I_\omega^{(M)}(\theta,\phi). \tag{2}$$

In this equation $\hat{\mathbf{k}}$ is the unit vector in the direction of the ray, $I_\omega^{(M)}(\theta,\phi)$, $j_\omega^{(M)}(\theta,\phi)$, and $\alpha_\omega^{(M)}(\theta,\phi)$ are the intensity, the emission coefficient, and the absorption coefficient, respectively, for the $M=O$, X mode of propagation of the wave. Here ω is the wave frequency, θ is the angle between the wave vector and the magnetic field (wave pitch angle), and ϕ is the azimuthal angle of propagation. It should be noted here that in some of the formulas the arguments and indices are omitted in order to obtain a shorter notation.

The operator $\hat{L}_{cy}$ includes both the radiation friction force due to spontaneous emission and the quasilinear diffusion due to stimulated emission, as given by

$$\hat{L}_{cy}(I,f) = \frac{4\pi^2 e^2}{m^2 c^3} \sum_{n=-\infty}^{\infty} \int dK\, \hat{L}_n \frac{\gamma}{\omega} D_n^{(M)} \delta\left(\gamma - \frac{n\omega_{ce}}{\omega} - u_\parallel \cos\theta\right) [I_\omega^{(M)}(\theta,\phi)\hat{L}_n + I_{RJ}\mu] f, \tag{3}$$

where $\gamma = \sqrt{1+u^2}$ is the relativistic factor, ω_{ce} is the cyclotron frequency for a particle at rest, and $\mu = mc^2/T$ is the inverse temperature.

The momentum space coordinates $\mathbf{p}$ are changed to the dimensionless $\mathbf{u}$ according to

$$\mathbf{p} = \mathbf{u} mc, \tag{4}$$

where $u_\perp$ and $u_\parallel$ are the transverse and parallel components, respectively.

The operator $\hat{L}_n$ is defined as

$$\hat{L}_n = \frac{n\omega_{ce}}{\omega u_\perp} \frac{\partial}{\partial u_\perp} + \cos\theta \frac{\partial}{\partial u_\parallel}. \tag{5}$$

The polarization factors are given as

$$D_n^{(X)} = \left(\frac{u_\perp}{\gamma} J_n'(\lambda)\right)^2, \quad D_n^{(O)} = \left(\frac{\gamma\cos\theta - u_\parallel}{\gamma\sin\theta} J_n(\lambda)\right)^2, \tag{6}$$

with J and J' denoting the Bessel function and its derivative, respectively, with its argument λ being

$$\lambda = \frac{n u_\perp \sin\theta}{\gamma - u_\parallel \cos\theta}. \tag{7}$$

The blackbody radiation intensity in the Rayleigh–Jeans limit, entering in Eq. (3), is given as

$$I_{RJ} = \frac{\omega^2 T}{8\pi^3 c^2}. \tag{8}$$

The absorption and emission coefficients in Eq. (2) have the form

$$\left\{\begin{array}{l}\alpha_\omega^{(M)}(\theta,\phi)\\ j_\omega^{(M)}(\theta,\phi)\end{array}\right\}=\left\{\begin{array}{l}\hat{\alpha}(f)\\ j(f)\end{array}\right\}$$
$$=\frac{4\pi^2e^2}{mc}\int dP\,\frac{\gamma}{\omega}\,D_n^{(M)}$$
$$\times\delta\left(\gamma-\frac{n\omega_{ce}}{\omega}-u_\parallel\cos\theta\right)\left\{\begin{array}{c}-\hat{L}_n\\ I_{\rm RJ}\mu\end{array}\right\}f. \tag{9}$$

It should be noted here that, e.g., the notation $\alpha_\omega^{(M)}(\theta,\phi)$ means that α is a function of ω, θ, and ϕ, whereas the notation $\hat{\alpha}(f)$ indicates that α is a functional of f.

Throughout the paper the following short notation for electron and photon phase space integration is used:

$$\int dP\equiv\sum_{n=-\infty}^{\infty}\int d^3p$$
$$=2\pi m^3c^3\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}du_\parallel\int_0^\infty du_\perp u_\perp, \tag{10}$$

$$\int dK\equiv\sum_{M=O,X}\int_0^\infty d\omega\int_{4\pi}d^2\Omega$$
$$=\sum_{M=o,x}\int_0^\infty d\omega\int_{-\pi}^{\pi}d\phi\int_0^\pi d\theta\sin\theta. \tag{11}$$

III. LINEARIZATION

For the purpose of linearization, the electron distribution function is presented in the form

$$f=f_0+\delta f, \tag{12}$$

where

$$f_0=\frac{n_e\mu}{4\pi m^3c^3K_2(\mu)}\exp(-\mu\gamma) \tag{13}$$

is the unperturbed Maxwellian distribution, and δf is a small perturbation. The radiation intensity is split into two parts:

$$I(\alpha,j)=I_0+\delta I(\delta\alpha,\delta j), \tag{14}$$

where I_0 is the unperturbed radiation intensity that satisfies the equation

$$\mathrm{div}(\hat{\mathbf{k}}I_0)=j_0-\alpha_0I_0, \tag{15}$$

where the unperturbed absorption and emission coefficients given by

$$\alpha_0=\hat{\alpha}(f_0),\quad j_0=\hat{j}(f_0)=\alpha_0I_{\rm RJ}, \tag{16}$$

are related through Kirchhoff's law. The perturbation of the intensity, δI, is caused by the deviation of the electron distribution function from the Maxwellian one. Here the arguments in the notation of the radiation intensities, the absorption, and emission coefficients, respectively, are omitted.

After substitution of Eqs. (12) and (14) into Eq. (1) one obtains the kinetic equation in the form

$$\hat{L}_{\rm C}\delta f+\hat{L}_{\rm EM}\,\delta f=-Q_{\rm EM}, \tag{17}$$

where $\hat{L}_{\rm C}$ is the linearized Coulomb collision integral.[16] It should be noted here, that the second-order term $(I-I_{\rm RJ})\delta f$ was neglected. The rest of the terms are regrouped in the following way:

$$\hat{L}_{\rm EM}\,\delta f\equiv\hat{L}_{\rm cy}(I_{\rm RJ},\delta f)+\hat{L}_{\rm cy}(I_{\rm RJ}+\delta I,f_0), \tag{18}$$

$$Q_{\rm EM}\equiv\hat{L}_{\rm cy}(I_0,f_0). \tag{19}$$

It should be noted that for the derivation of Eq. (17) one should use the obvious equality $\hat{L}_{\rm cy}(I_{\rm RJ},f_0)=0$.

Note that the subintegrand of the neglected term, $(I-I_{\rm RJ})\hat{L}_n\,\delta f$, is not always small compared to the retained one, $I_{\rm RJ}\hat{L}_n\,\delta f$, because the intensity I is sufficiently different from the Rayleigh–Jeans intensity $I_{\rm RJ}$ in the high-frequency region. For this reason Eq. (17) is not valid for a particle distribution function in the distant energy tail, which, due to the effect of the radiation friction force, differs strongly from the Maxwellian distribution function at those high energies (see Ref. 17). However, the particles giving the main contribution to the cyclotron losses and to the current drive, stay not very far away in the tail, and the corresponding integral is small compared to the retained one and can be safely neglected. This comes from the fact that most of the energy and momentum is radiated (and absorbed) by the particles being of interest in the low-frequency region below a certain cutoff frequency defined as a solution to the following equation:

$$a\alpha^{(M)}(\overline{\omega}_{\rm cut}^{(M)}(\theta),\theta)=1, \tag{20}$$

where a is the typical plasma size and $\overline{\omega}=\omega/\omega_{ce}$ is the dimensionless frequency. In this frequency region the plasma is optically thick and the radiation is absorbed almost locally, staying very close to the Rayleigh–Jeans law. With the same argument, in this region one can neglect in Eq. (2) the radiation transport term $\mathrm{div}(\hat{\mathbf{k}}\,\delta I)$ as well as the small term $\delta\alpha(I-I_{\rm RJ})$. Using Eq. (15) for δI one obtains the local relation

$$\delta j-\delta\alpha\,I_{\rm RJ}-\alpha_0\,\delta I=0, \tag{21}$$

where

$$\delta j=\hat{j}(\delta f),\quad \delta\alpha=\hat{\alpha}(\delta f). \tag{22}$$

Thus, the quantity $\hat{L}_{\rm EM}$ can be interpreted as the local linearized radiation collision operator. The only nonlocality of the problem is contained in Eq. (17) in the source function $Q_{\rm EM}$, which is almost the same as used in Ref. 2. It differs only because of the presence of a convective term corresponding to the radiation friction force. But this term is rather unimportant for the current drive problem under consideration, because of its symmetry over parallel momentum.

In order to verify the approximation, one has to make sure that the power fraction radiated at frequencies above the cutoff frequency is small compared to the total power emitted by the particles from the phase space region mainly responsible for the losses. It can be presented as

$$r_\epsilon=\frac{\sum_{M=O,X}\int_0^\pi d\theta\int_{\overline{\omega}_{\rm cut}^{(M)}(\theta)}^\infty d\overline{\omega}\,W^{(M)}(\overline{\omega},\theta)}{\sum_{M=O,X}\int_0^\pi d\theta\int_0^\infty d\overline{\omega}\,W^{(M)}(\overline{\omega},\theta)}, \tag{23}$$

where $W^{(M)}$ is the power density emitted at a given frequency and wave pitch angle by a single particle,

$$W^{(M)}(\overline{\omega},\theta)=\gamma \frac{e^2\omega_{ce}^2}{c}\sum_{n=1}^{\infty} \sin \theta \overline{\omega}^2 D_n^{(M)} \times \delta[n-(\gamma-u_\parallel \cos \theta)\overline{\omega}]. \tag{24}$$

Taking into account that most of the losses occur at wave pitch angles close to $\theta=\pi/2$ and that the corresponding particle pitch angles are also close to $\chi=\pi/2$, one can set the cutoff frequency to its value at $\theta=\pi/2$ and consider only the particles with $u_\parallel=0$ for estimating the power fraction, Eq. (23).

The procedure of making such an estimate is well known (e.g., see Ref. 17). One should take into account the overlapping of cyclotron resonances, which allows us to change the summation over the harmonics to an integration in Eq. (24). This integration can be performed by means of the δ function. After substitution of the asymptotical formulas for the Bessel functions that enter the polarization factors $D_n^{(M)}$, the integration over the wave pitch angle can be performed with a steepest decent method. The remaining integration over the frequency in the denominator in Eq. (23) is elementary. In the nominator one can use the fact that the main contribution just comes from the neighborhood of the cutoff region and perform the integration by parts. Finally for r_ϵ one obtains

$$r_\epsilon \approx \frac{4}{5}\sqrt{\frac{3}{2\pi}}\frac{1}{\gamma}\left[\overline{\omega}_X^{1/2} \exp\left(-\frac{2\overline{\omega}_X}{3\gamma^2}-\frac{2\overline{\omega}_X}{5\gamma^5}\right) + \frac{\gamma^2}{2\overline{\omega}_O^{1/2}} \exp\left(-\frac{2\overline{\omega}_O}{3\gamma^2}-\frac{2\overline{\omega}_O}{5\gamma^5}\right)\right], \tag{25}$$

where $\overline{\omega}_M \equiv \overline{\omega}_{\rm cut}^{(M)}(\pi/2)$.

The formula for the absorption coefficient is obtained in a similar way. The only difference is that the integration is performed in the electron momentum space. The last integration over energy can also be performed with a steepest decent method, giving (see Ref. 17)

$$\alpha_{\pi/2}^{(M)}=\frac{3}{2}\sqrt{\pi\mu}\frac{\omega_{pe}^2}{c\,\omega_{ce}}\frac{1}{\kappa}D_{\pi/2}^{(M)} \times \exp\left[-\mu\left(\kappa^{1/3}-1+\frac{9}{20\kappa^{1/3}}\right)\right], \tag{26}$$

with $D_{\pi/2}^{(X)}=1$ and $D_{\pi/2}^{(O)}=1/(\mu\kappa^{1/3})$. Here

$$\kappa=\frac{9\overline{\omega}}{2\mu}=\frac{27}{8}\gamma_0^3 \tag{27}$$

is determined by the position of the saddle point on the energy axis. This also provides one with the correspondence between the wave frequency and the energy of the particles responsible for the emission and absorption at this frequency,

$$\gamma_0=\left(\frac{4}{3}\frac{\overline{\omega}}{\mu}\right)^{1/3}. \tag{28}$$

Using the approximate formula (26), the equation for the cutoff frequency (20) can be written as

$$1=\Lambda\frac{1}{\kappa_X}\exp\left[-\mu\left(\kappa_X^{1/3}+\frac{9}{20\kappa_X^{1/3}}\right)\right], \tag{29}$$

$$1=\Lambda\frac{1}{\kappa_O}\frac{1}{\mu\kappa_O^{1/3}}\exp\left[-\mu\left(\kappa_O^{1/3}+\frac{9}{20\kappa_O^{1/3}}\right)\right], \tag{30}$$

where

$$\Lambda=\frac{3}{2}\sqrt{\pi\mu}e^\mu\frac{\omega_{pe}^2}{c\,\omega_{ce}}a \gg 1. \tag{31}$$

Since $\mu\kappa_O^{1/3}\ll\Lambda$ one can write $\kappa_O=\kappa_X+\Delta\kappa$, where $\Delta\kappa/\kappa_X\ll 1$. Taking into account Eq. (29), one obtains

$$\frac{\Delta\kappa}{\kappa_X}=-\frac{3\ln(\mu\kappa_X^{1/3})}{\mu\kappa_X^{1/3}}=\frac{\overline{\omega}_O-\overline{\omega}_X}{\overline{\omega}_O}. \tag{32}$$

To find out which particles are the most dangerous ones for the approximation one has to find the "highest" frequency, still giving a sufficient contribution to the losses from the equation

$$A=a\alpha_{\pi/2}^{(X)}=\frac{\Lambda}{\kappa_{UX}}\exp\left[-\mu\left(\kappa_{UX}^{1/3}+\frac{9}{20\kappa_{UX}^{1/3}}\right)\right], \tag{33}$$

where $A=1-\Gamma$ is the absorption coefficient at the wall. With $A^{-1}\ll\Lambda$ one can use the same approach as before to obtain

$$\frac{\Delta\kappa_U}{\kappa_X}=-\frac{3\ln A}{\mu\kappa_X^{1/3}}, \tag{34}$$

where $\kappa_{UX}=\kappa_X+\Delta\kappa_U$ is the κ value for this upper frequency of the extraordinary mode. Using Eq. (34) one can express the relativistic factor γ of the considered particles through

$$\gamma=\gamma_X\left(1+\frac{\Delta\kappa_U}{3\kappa_X}\right). \tag{35}$$

Here γ_X is given by Eq. (28) with the substitution of the cutoff frequency of the extraordinary mode.

Expressing in Eq. (25), the quantities γ, $\overline{\omega}_O$, and $\overline{\omega}_X$ through the quantity κ_X and using Eq. (29), for the power fraction, one finally obtains

$$r_\epsilon \approx \frac{4\cdot 3^{1/2}}{5\pi^{1/2}}\frac{(\ln\Lambda)^{3/2}}{\mu\Lambda^{1/3}A^{2/3}}\exp\left(-\frac{3\mu^2}{10\ln\Lambda}\right). \tag{36}$$

As soon as this parameter r_ϵ is small, the linearized system of Eqs. (15) and (17) stays to be valid for the problems of cyclotron losses and passive cyclotron current drive, even without a sufficient Coulomb relaxation. In Fig. 1 the behavior of the power fraction r_ϵ is shown for a typical plasma with the density $n_e=2\times10^{19}$ m^{-3}, $\mu=6$, and the stationary magnetic field $B_0=10$ T for different values of the plasma size a. One can see that r_ϵ is more restrictive at high values of wall reflectivity staying, however, small for realistic reflectivity values.

IV. RADIATION COLLISION OPERATOR

Introducing the normalized perturbed distribution function $\hat{f}$ as

$$\hat{f}\equiv\frac{\delta f}{f_0}, \tag{37}$$

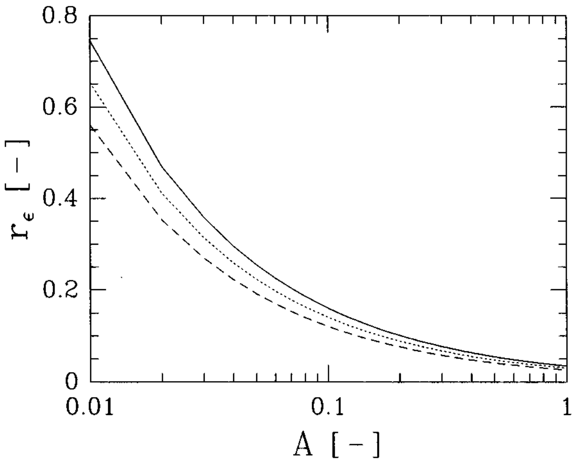

FIG. 1. Quantity r_ϵ versus the wall absorption coefficient for different values of the plasma size $a=0.3$ m (solid line); $a=1.0$ m (dotted line); and $a=3.0$ m (dashed line).

and substituting the perturbed intensity δI from Eq. (21), for the linearized radiation collision operator, one explicitly obtains

$$\hat{L}_{\rm EM}\ \delta f=\frac{4\pi^2e^2}{m^2c^3}\sum_{n=-\infty}^{\infty}\int dK\ \hat{L}_nA \times\left[\hat{L}_n\hat{f}-\left(\int dP\ A\hat{L}_n\hat{f}\ \right)\bigg/\left(\int dP\ A\right)\right], \tag{38}$$

with the following notation for A

$$A\equiv A_n^{(M)}(u_\perp,u_\parallel,\omega,\theta) \equiv D_n^{(M)}\ \frac{\gamma}{\omega}\ \delta\left(\gamma-\frac{n\omega_{ce}}{\omega}-u_\parallel\cos\theta\right)f_0I_{\rm RJ}. \tag{39}$$

This operator preserves the parallel momentum and the total energy, which can be shown by multiplying Eq. (38) with either γ or $u_\parallel$, and subsequent integration over the momentum space. The parity of the distribution function over the change $u_\parallel$ to $-u_\parallel$ is preserved as well.

Let us introduce now a variational principle similar to the one derived in Refs. 13 and 14 for the linearized Coulomb collision integral. Multiplying Eq. (38) by $\hat{g}$ and integrating over the electron momentum space, one obtains the positively definite quadratic integral functional,

$$K_{\rm EM}(\hat{g},\hat{f}\ )\equiv-\int d^3p\ \hat{g}\hat{L}_{\rm EM}\ \delta f =\frac{4\pi^2e^2}{m^2c^3}\int dK\left[\int dP\ A(\hat{L}_n\hat{g})(\hat{L}_n\hat{f}) -\left(\int dP\ A(\hat{L}_n\hat{g})\right)\left(\int dP\ A(\hat{L}_n\hat{f})\right)\bigg/ \left(\int dP\ A\right)\right]. \tag{40}$$

One of the properties of the radiation collision operator is its self-adjointness with respect to the same scalar product as the Coulomb collision operator. This follows from the symmetry of the functional $K_{\rm EM}$,

$$K_{\rm EM}(\hat{g},\hat{f})=K_{\rm EM}(\hat{f},\hat{g}). \tag{41}$$

The positive definiteness of $K_{\rm EM}$ follows from the Boltzmann H theorem, which can be checked by computing the local rate of entropy production (see, e.g., Ref. 14),

$$\frac{dS}{dt}\equiv K_{\rm EM}(\hat{f},\hat{f}\ ) =4\pi^2e^2m^2c\int dK\int dP \times A\left[\hat{L}_n\hat{f}-\left(\int dP\ A(\hat{L}_n\hat{f}\ )\right)\bigg/\left(\int dP\ A\right)\right]^2. \tag{42}$$

The equilibrium distribution is reached when the entropy production stops. As one can see from Eq. (5), the operator $\hat{L}_n$ has the following form:

$$\hat{L}_n=\frac{1}{u_\perp}\ (\gamma-u_\parallel\cos\theta)\ \frac{\partial}{\partial u_\perp}+\cos\ \theta\ \frac{\partial}{\partial u_\parallel} =\frac{\partial}{\partial\gamma}+\cos\ \theta\ \frac{\partial}{\partial u_\parallel}. \tag{43}$$

This operator actually does not depend on the index n and the frequency (due to the δ function), so the right-hand side of Eq. (42) is zero if $\hat{L}_n\hat{f}$ is constant everywhere in the electron space. But this should be true for all θ angles (due to integration over them), so the derivatives of $\hat{f}$ over γ and $u_\parallel$ should be separate constants. This gives one the unique form of the equilibrium distribution

$$\hat{f}=C_0+C_\gamma\gamma+C_pu_\parallel, \tag{44}$$

where all three constants are much smaller than unity, because of the use of the linearized collision integral. One can easily see that Eq. (44) is the linear term in the expansion of a shifted Maxwellian distribution with a modified density and temperature:

$$f_0^{\rm sh}={\rm const}\ \exp\left(-\frac{mc^2\gamma-V_\parallel p_\parallel}{(T+\Delta T)\sqrt{1-(V_\parallel/c)^2}}\right), \tag{45}$$

where $V_\parallel$ is the velocity of a frame moving along the magnetic field line. In such a moving frame the given Maxwellian would be at rest.

The linearized radiation collision operator (38) can be presented in an invariant form,

$$\left(\frac{\partial\delta f}{\partial t}\right)_{\rm EM}=\frac{1}{J}\ \frac{\partial}{\partial y^i}\ Jf_0\left(D^{ij}\ \frac{\partial\hat{f}}{\partial y^j} -\int d^3p'\ f_0'R^{ij}(\mathbf{y},\mathbf{y}')\ \frac{\partial\hat{f}'}{\partial y'^j}\right), \tag{46}$$

where prime denotes the dependence on a primed argument, $y^i\equiv\mathbf{y}$ are the coordinates in the electron momentum space, and $J=\partial(\mathbf{p})/\partial(\mathbf{y})$ is the Jacobian of the coordinate system. In the case considered above $\mathbf{y}=(u_\perp, u_\parallel)$ and $J=u_\perp m^3 c^3$. Here the gyrophase ϕ was omitted because nothing depends on it. Using Eq. (46) for the bilinear form (40) one can obtain

$$K_{\rm EM}(\hat{g},\hat{f}\,)=\int d^3p\; f_0\,\frac{\partial\hat{g}}{\partial y^i}\,D^{ij}\,\frac{\partial\hat{f}}{\partial y^j} - \int d^3p\int d^3p'\; f_0 f_0'\,\frac{\partial\hat{g}}{\partial y^i}\,R^{ij}(\mathbf{y},\mathbf{y}')\,\frac{\partial\hat{f}'}{\partial y'^j}, \tag{47}$$

where integration over d^3p means the general form $d^3y\,J$. Performing the integration over the photon phase space variables in Eq. (40), one can obtain both the diffusion tensor and the kernel of the integral part in Eq. (47).

The shortest form of the diffusion tensor can be obtained after the summation over cyclotron harmonics n in a coordinate system using the pitch angle $\chi=\arctan(p_\perp/p_\parallel)$ and the momentum module $p=\sqrt{p_\perp^2+p_\parallel^2}$ as independent variables:

$$D^{pp}=\frac{2e^2\omega_{ce}^2 T}{3c^3}\,\gamma^2\sin^2\chi, \tag{48}$$

$$D^{p\chi}=D^{\chi p}=\frac{2e^2\omega_{ce}^2 T}{3c^3 p}\sin\chi\cos\chi, \tag{49}$$

$$D^{\chi\chi}=\frac{e^2\omega_{ce}^2 T}{24c^3p^2}\left\{10+\frac{3}{u^2}-\left(\frac{9}{\gamma u}+\frac{3\gamma}{u^3}\right)\ln(u+\gamma) - \left[10+\frac{15}{u^2}-\left(\frac{21}{u\gamma}+\frac{15\gamma}{u^3}\right)\ln(u+\gamma)\right]\cos^2\chi\right\}, \tag{50}$$

where u is the normalized momentum module [see Eq. (4)].

The most convenient coordinate system for presenting the kernel in Eq. (47) is in a coordinate system using γ and $u_\parallel$ with the Jacobian $J=m^3c^3\gamma$, resulting in

$$\begin{pmatrix} R^{\gamma\gamma'} & R^{\gamma u_\parallel'} \\ R^{u_\parallel\gamma'} & R^{u_\parallel u_\parallel'}\end{pmatrix}=\frac{2e^2\omega_{ce}^2K_2(\mu)}{mc^3n_e\mu^2}\cdot\frac{\gamma\gamma'}{|\gamma u_\parallel'-\gamma' u_\parallel|^3}\sum_{M=0,X}\sum_{n,n'=1}^{\infty}D_n^{(M)}D_{n'}'^{(M)}\Theta(1-|\lambda_0|)|nu_\parallel'-n'u_\parallel|\cdot|n\gamma'-n'\gamma|\begin{pmatrix}1 & \lambda_0\\ \lambda_0 & \lambda_0^2\end{pmatrix} \times\left(\sum_{n''=1}^{\infty}\int_1^\infty d\gamma''\;\gamma''^2e^{-\mu\gamma''}\,D_{n''}''^{(M)}\Theta(\gamma''^2-u_\parallel''^2-1)\right)^{-1}. \tag{51}$$

Here

$$\lambda_0=\frac{n\gamma'-n'\gamma}{nu_\parallel'-n'u_\parallel}, \tag{52}$$

$$u_\parallel''=\frac{u_\parallel'(n\gamma''-n''\gamma)-u_\parallel(n'\gamma''-n''\gamma')}{n\gamma'-n'\gamma}, \tag{53}$$

where for the polarization factors one has to use Eq. (6) with $u_\perp=(\gamma^2-u_\parallel^2-1)^{1/2}$, $u_\parallel$ and γ with the number of primes (0,1,2) corresponding to the number of primes on the polarization factor. The wave pitch angle θ entering the polarization factors given by Eq. (6), should be taken as $\theta=\arccos\lambda_0$.

Using the differential part of the form (38) and the detailed equilibrium principle (each component of the flux is zero separately for the Maxwellian distribution $\hat{f}=$const), one can obtain the expression for the radiation friction force as

$$F^i=D^{ij}\,\frac{\partial\ln f_0}{\partial y^j}=-\mu D^{ij}\,\frac{\partial\gamma}{\partial y^j}. \tag{54}$$

Using Eq. (48) in the initial variables $p_\perp, p_\parallel$, one obtains

$$F^\perp=-\frac{2e^2\omega_{ce}^2}{3mc^3}\,\frac{p_\perp}{\gamma}\left[1+\left(\frac{p_\perp}{mc}\right)^2\right],$$

$$F^\parallel=-\frac{2e^2\omega_{ce}^2}{3mc^3}\,\frac{p_\parallel}{\gamma}\left(\frac{p_\perp}{mc}\right)^2, \tag{55}$$

which is the same as obtained with direct calculation.[18]

V. PASSIVE CYCLOTRON CURRENT DRIVE EFFICIENCY

At this point, the problem of finding the passive current drive efficiency in a uniform magnetic field can be formulated in a variational form as an adjoint problem of finding the generalized Spitzer–Härm distribution function.[15,19,20] Although this procedure is well developed for the case of Coulomb collisions being the only relaxation mechanism, we briefly repeat this formulation; which now also includes the collisions due to radiation. This is necessary mainly to clarify the notation, which is further used, and to introduce the passive cyclotron current drive efficiency, which is different from the one normally used in the literature. The solution to the problem of finding the minimum of the functional,

$$K_p(\hat{f}\,)\equiv K(\hat{f},\hat{f})-2\int d^3p\;\hat{f}Q_{\rm EM}, \tag{56}$$

satisfies at the same time the linearized kinetic equation (17). Here $\hat{f}$ is the normalized perturbed distribution function (37) and K is a positively definite quadratic functional symmetric over its arguments,

$$K(\hat{g},\hat{f}\,)\equiv K_{\rm C}(\hat{g},\hat{f})+K_{\rm EM}(\hat{g},\hat{f}\,). \tag{57}$$

Here $K_{\rm EM}$ is given by Eq. (47). The corresponding functional $K_{\rm C}$ for Coulomb collisions is defined as

$$K_{\rm C}(\hat{g},\hat{f})\equiv-\int d^3p\;\hat{g}\hat{L}_C\;\delta f=K_C^{(e)}(\hat{g},\hat{f})+K_C^{(i)}(\hat{g},\hat{f}). \tag{58}$$

where the superscripts e and i denote electron–electron and electron–ion collisions, respectively. Explicitly, one has

$$K_C^{(\alpha)}(\hat{g},\hat{f}\,)=\int d^3p\; D_\alpha^{ij}(\mathbf{y})f_0(\mathbf{y})\,\frac{\partial\hat{g}(\mathbf{y})}{\partial y^i}\,\frac{\partial\hat{f}(\mathbf{y})}{\partial y^j}-\int d^3p\int d^3p'\; f_0(\mathbf{y})f_0(\mathbf{y}')R_\alpha(\mathbf{y},\mathbf{y}')\times\hat{g}(\mathbf{y})\hat{f}(\mathbf{y}'). \tag{59}$$

As given by Ref. 16, the nonzero components of the diffusion tensor for the electron–electron collisions have the following form in p, χ variables:

$$D_e^{pp}(p)=\frac{\pi m^4c^3\Gamma\gamma}{n_ep^3}\left(\int_0^p dp'\,\frac{p'}{\gamma'}\,f_0(p')[2\gamma^2(u'\gamma'-\sigma')+3u'\gamma'-(3+2u'^2)\sigma']+\int_p^\infty dp'\,\frac{p'}{\gamma'}\,f_0(p')\times[2\gamma'^2(u\gamma-\sigma)+3u\gamma-(3+2u^2)\sigma]\right),$$

$$D_e^{\chi\chi}(p)=\frac{\pi m^2c\Gamma\gamma}{n_ep^3}\left\{\int_0^p dp'\,\frac{p'}{\gamma'}\,f_0(p')\left[2u'\gamma'-\left(\frac{1}{u^2}+\frac{1}{\gamma^2}\right)(u'\gamma'-\sigma')+\frac{1}{2u^2\gamma^2}\,[(3+2u'^2)\sigma'-3u'\gamma']\right]+\int_p^\infty dp'\,\frac{p'}{\gamma'}\,f_0(p')\left(2\gamma'^2\,\frac{u}{\gamma}-u'^2\,\frac{1}{u^2\gamma^2}\,(u\gamma-\sigma)+\frac{1}{2u^2\gamma^2}\,[(1+4\gamma^2)\sigma-(3+2\gamma^2)u\gamma]\right)\right\}. \tag{60}$$

The kernel of the integral part of the electron–electron collision operator has the form

$$R_e(\mathbf{y},\mathbf{y}')=\cos\chi\,\cos\chi'[R_1(p,p')+R_2(p,p')+R_2(p',p)], \tag{61}$$

where

$$R_1(p,p')=\frac{3\Gamma m}{n_e\gamma p^2}\,\delta(p-p'), \tag{62}$$

$$R_2(p,p')=\frac{3\Gamma m^2c^2}{n_ep^2p'^2\gamma\gamma'}\left[u'\gamma'-\sigma'+\frac{\mu}{2}\,(5\gamma'\sigma'-5u'-u'^3)+\gamma\left(\frac{\mu}{2}\,[(5+2u'^2)\sigma'-5u'\gamma']+\frac{\mu^2}{12}\,[3u'-5u'^3-(33+6u'^2)\gamma'\sigma']\right)+u^2\mu(\gamma'\sigma'-u')+u^2\gamma\,\frac{\mu^2}{6}\,(3u'+u'^3-3\gamma'\sigma')\right]\Theta(p-p'), \tag{63}$$

and

$$\Gamma=4\pi n_ee^4\Lambda_{ee}. \tag{64}$$

In the formulas (60) and (62) $u=p/mc$, $u'=p'/mc$, $\gamma=\sqrt{1+u^2}$, $\gamma'=\sqrt{1+u'^2}$, $\sigma=\ln(u+\gamma)$, and $\sigma'=\ln(u'+\gamma')$. The electron–ion collisions are well described by the Lorentz model because the only important effect is pitch-angle scattering represented by the diffusion tensor component $D_i^{\chi\chi}$, given as

$$D_i^{\chi\chi}(p)=\frac{\Gamma Z_im\gamma}{2p^3}. \tag{65}$$

In the integral part of the collision operator (61), which is derived in its full form in Refs. 21 and 16, only the first harmonic of the Legendre expansion of the kernel over the pitch-angle variable is retained. This, however, permits us to solve the current drive efficiency problem exactly for the case of a uniform magnetic field with Coulomb collisions being the only relaxation mechanism. Also, it continues to be a good approximation in the case of a nonuniform magnetic field (see Ref. 20) or in the case where collisions due to radiation are taken into account.

Putting the variation of the functional (56) over $\hat{f}$ to zero, one obtains the kinetic equation (17). Therefore, the minimum function of the functional (56) is the general solution for the electron equilibrium problem. However, for the current drive problem the general solution is not necessary. The only quantity of interest is the parallel current density,

$$j_\parallel=e\int d^3p\;\delta fv_\parallel=\frac{e}{m}\int d^3p\;\frac{p\,\cos\chi}{\gamma}\,f_0(p)\hat{f}. \tag{66}$$

Following Refs. 15 and 19 one can present this quantity in the form

$$j_\parallel=\int d^3p\;\hat{g}Q_{\mathrm{EM}}, \tag{67}$$

where the reduced Green's-function of the full collision operator $\hat{g}$ satisfies the generalized Spitzer–Härm equation,

$$\hat{L}_C(f_0\hat{g})+\hat{L}_{\mathrm{EM}}(f_0\hat{g})=-ev_\parallel f_0. \tag{68}$$

This property follows from the symmetry of the functional (57) over its arguments. Equation (68) is equivalent to the minimum condition of the functional $K_{\mathrm{SH}}(\hat{g})$ given by

$$K_{\mathrm{SH}}(\hat{g})=K(\hat{g},\hat{g})-2\int d^3p\;ev_\parallel f_0\hat{g}. \tag{69}$$

Due to the generality of the function $\hat{g}$, one can obtain the parallel current for any linear current drive method with power input to the electrons without the explicit solution of the kinetic equation. However, for the problem considered here, it is more convenient to introduce the passive cyclotron current drive efficiency defined as

$$\eta_\omega^{(M)}(\theta)=\frac{4\pi^2e^2}{m^2c^3}\,\frac{\mu}{I_{\mathrm{RJ}}}\int dP\;A\hat{L}_n\hat{g}, \tag{70}$$

with A and $\hat{L}_n$ given by Eqs. (39) and (5), respectively. This quantity links the parallel current density directly to the radiation intensity through

$$j_{\parallel}=\int dK\ \eta_{\omega}^{(M)}(\theta)I_{\omega}^{(M)}(\theta,\phi), \tag{71}$$

thus giving the amount of parallel current produced by the unit amount of radiation from the unit interval of frequencies and angles. In this sense, it is completely analogous to the absorption coefficient (9). The only difference is that it gives the current density instead of the absorbed power density.

VI. CURRENT DRIVE EFFICIENCY IN TOROIDAL GEOMETRY

In order to describe electron kinetics in tokamak geometry, the set of guiding center variables (x^i,y^i) is introduced. The momentum space variables $y^i=(p_0,\chi_0,\phi)$ are the momentum module, the pitch angle in the magnetic field minimum on the given magnetic surface, and the gyrophase, respectively. The first two variables are the integrals of motion expressed through the local values of the momentum module and the pitch angle,

$$p=p_0, \quad \frac{\sin^2\chi}{B_0}=\frac{\sin^2\chi_0}{B_{\min}}, \tag{72}$$

where B_0 is the local value of the magnetic field and $B_{\min}$ is its minimum value on the given surface. The spatial guiding center variables $x^i=(\Psi,\vartheta,\zeta)$ are the normalized poloidal flux and the poloidal and toroidal angles of the quasitoroidal coordinate system (r,ϑ,ζ), respectively. For the poloidal flux one has the expression

$$\Psi=\int_0^r dr'\ R(r',\vartheta)B_{\vartheta}(r',\vartheta)=\Psi(r,\vartheta), \tag{73}$$

where $R=R_0+r\cos\vartheta$ is the actual value of the big radius, R_0 is its value for the magnetic axis, and B_{ϑ} is the poloidal physical magnetic field component in quasitoroidal variables. The Jacobians of momentum space and coordinate space variables are given as

$$J_y=p^2\sin\chi_0\,\frac{B_0}{B_{\min}}\,\frac{\cos\chi_0}{\cos\chi}, \quad J_x=\frac{r}{B_{\vartheta}}. \tag{74}$$

Neglecting the drift motion, the kinetic equation can be cast to the form

$$v^{\vartheta}\,\frac{\partial\delta f}{\partial\vartheta}=\hat{L}_{\mathrm{C}}\ \delta f+\hat{L}_{\mathrm{EM}}\ \delta f+Q_{\mathrm{EM}}. \tag{75}$$

Because of the toroidal symmetry, only the poloidal variation of the distribution function is taken into account in Eq. (75), where

$$v^{\vartheta}=\frac{d\vartheta}{dt}=\frac{p}{m\gamma}\,\frac{B_{\vartheta}\cos\chi}{B_0 r}$$
$$=\frac{p_0}{m\gamma}\cos\chi_0\,\frac{B_{\vartheta}}{rB_0}\,\sqrt{1+\left(1-\frac{B_0}{B_{\min}}\right)\tan^2\chi_0} \tag{76}$$

is the poloidal velocity. The collision operators on the right-hand side of Eq. (75) have a covariant form and can be easily transformed to the new set of variables x^i,y^i according to the rules of tensor algebra. Small terms appear in such a transform containing the derivatives over the guiding center position. They have the order of the ratio of the Larmor radius to the magnetic field scale and can be safely neglected.

The derivation of Eq. (75) is valid if the collision operators, as well as the source term Q_{EM} obtained for the case of the uniform magnetic field, are locally valid for the nonuniform magnetic field case. The local formula for the electromagnetic interaction operator (3) in turn is valid if the correlation time of the radiation electromagnetic field Δt_c, which limits the process of wave–particle interaction, is short enough compared to the time of wave–particle phase change due to the magnetic field nonuniformity along the particle trajectory $\Delta t_B=(v_{\parallel}n\ d\omega_c/ds)^{-1/2}$. Here d/ds means the derivative along the magnetic field line and n is the characteristic number of the cyclotron resonance. The correlation time is determined mainly by the Doppler broadening of resonances, which gives $\Delta t_c=(v_{\parallel}n\omega_c/c)^{-1}$. It is usually small enough compared to Δt_B in tokamaks with realistic parameters for the plasma and the magnetic field.

In the ''collisionless'' confinement regime $\nu_c\ll v^{\vartheta}$ (ν_c is the collision frequency), Eq. (75) can be simplified with the help of the bounce-averaging procedure. The same result in a shorter way can be obtained by averaging the functional (56) over the volume between two neighboring magnetic surfaces (see Ref. 14). In order to obtain such an averaged functional, one has to transform the local momentum space integration variables to the set of new variables y^i introduced above. In particular, the functional (57) entering Eq. (56) after this transformation takes the general form

$$K(\hat{g},\hat{f})=\int d^3y\ J_y f_0 D^{ij}\,\frac{\partial\hat{g}}{\partial y^i}\,\frac{\partial\hat{f}}{\partial y^j}$$
$$-\int d^3y\int d^3y'\ J_y J_y' f_0 f_0' R^{ij}(\mathbf{y},\mathbf{y}')\,\frac{\partial\hat{g}}{\partial y^i}\,\frac{\partial\hat{f}}{\partial y'^j}$$
$$-\int d^3y\int d^3y'\ J_y J_y' f_0 f_0' R(\mathbf{y},\mathbf{y}')\hat{g}(\mathbf{y})\hat{f}(\mathbf{y}'). \tag{77}$$

It should be noted, here, that D^{ij}, R^{ij} ($\mathbf{y}$, $\mathbf{y}'$) and $R(\mathbf{y},\ \mathbf{y}')$ contain both Coulomb collisions and collisions due to radiation expressed in the new variables y^i. All the primed quantities in Eq. (77) are the functions of the primed argument $\mathbf{y}'$. The average over the layer between two neighboring magnetic surfaces is introduced as

$$\langle K\rangle=\left(\int_{-\pi}^{\pi}d\vartheta\ J_x(\Psi,\vartheta)\right)^{-1}$$
$$\times\int_{-\pi}^{\pi}d\vartheta\ J_x(\Psi,\vartheta)K(\Psi,\vartheta), \tag{78}$$

which is also referred to as the poloidal average. Neglecting small corrections to the distribution functions $\hat{f}$ and $\hat{g}$, which come from the dependence on ϑ, and assuming these functions to be continuous and symmetric over $\cos\kappa_0$ in the region of trapped particles $\pi-\kappa_b<\kappa_0<\kappa_b$, where $\tan^2\kappa_b=B_{\min}/(B_{\max}-B_{\min})$ and $B_{\max}$ is the maximum value of the magnetic field on a given surface, one obtains for the average of the functional (77),

$$\langle K(\hat{g},\hat{f})\rangle = \int d^3y\ J_b f_0 \langle D^{ij}\rangle_b \frac{\partial \hat{g}}{\partial y^i}\frac{\partial \hat{f}}{\partial y^j} - \int d^3y \int d^3y'\ J_b J_b' f_0 f_0' \langle R^{ij}(\mathbf{y},\mathbf{y}')\rangle_b \times \frac{\partial \hat{g}}{\partial y^i}\frac{\partial \hat{f}}{\partial y^j} - \int d^3y \int d^3y'\ J_b J_b' f_0 f_0' \times \langle R(\mathbf{y},\mathbf{y}')\rangle_b \hat{g}(\mathbf{y})\hat{f}(\mathbf{y}'). \tag{79}$$

Here the quantity J_b is defined as

$$J_b = \langle J_y \rangle, \tag{80}$$

and the bounce-averaged quantities, which are functions of one argument in the momentum space $\mathbf{y}$, such as the diffusion coefficients, are given by

$$\langle D^{ij}\rangle_b = \frac{1}{J_b}\langle J_y D^{ij}\rangle. \tag{81}$$

For the bounce-averaged kernels of the integral part of the collision operators, one obtains the expressions

$$\langle R^{ij}(\mathbf{y},\mathbf{y}')\rangle_b = \frac{1}{J_b(\mathbf{y})J_b(\mathbf{y}')}\langle J_y(\mathbf{y})J_y(\mathbf{y}')R^{ij}(\mathbf{y},\mathbf{y}')\rangle \tag{82}$$

and

$$\langle R\rangle_b = \frac{1}{J_b(\mathbf{y})J_b(\mathbf{y}')}\langle J_y(\mathbf{y})J_y(\mathbf{y}')R(\mathbf{y},\mathbf{y}')\rangle. \tag{83}$$

Formulas (80)–(83) correspond to the region of passing particles in the phase space, which is important for the current drive problem.

Finally the averaged functional (56) takes the form

$$\langle K_p(\hat{f})\rangle = \langle K(\hat{f},\hat{f})\rangle - 2\int d^3y\ J_b \hat{f}\langle Q_{\rm EM}\rangle_b. \tag{84}$$

Here the bounce-averaged source $\langle Q_{\rm EM}\rangle_b$ is obtained in the same way as the bounce-averaged diffusion tensor (81). The zero variation of this functional corresponds to the bounce-averaged equation. However, as in the previous section, we are interested only in the toroidal current density averaged over the meridian plane area between the cross sections of two neighboring magnetic surfaces, as given by

$$\langle j_t\rangle_S = \frac{j_\parallel}{B_0}\left\langle \frac{B_\zeta}{R}\right\rangle \left\langle \frac{1}{R}\right\rangle^{-1}. \tag{85}$$

This quantity is expressed through the normalized perturbed distribution function $\hat{f}$ as

$$\langle j_t\rangle_S = \int d^3y\ J_b f_0 \hat{f}\langle W\rangle_b, \tag{86}$$

where

$$\langle W\rangle_b = e\left\langle \frac{v_\parallel}{B_0}\right\rangle_b \left\langle \frac{B_\zeta}{R}\right\rangle \left\langle \frac{1}{R}\right\rangle^{-1} = \frac{e p_0^3 \sin\chi_0 \cos\chi_0}{J_b B_{\min} m\gamma_0}\left\langle \frac{B_\zeta}{R}\right\rangle \left\langle \frac{1}{R}\right\rangle^{-1}. \tag{87}$$

Introducing the generalized Spitzer–Härm function $\hat{g}$ that corresponds to the minimum of the functional

$$K_W(\hat{g}) \equiv \langle K(\hat{g},\hat{g})\rangle_b - 2\int d^3y\ J_b f_0 \langle W\rangle_b \hat{g}, \tag{88}$$

for the averaged toroidal current density, one has the expression (see also Refs. 15 and 19)

$$\langle j_t\rangle_S = \int d^3y\ J_b \hat{g}\langle Q_{\rm EM}\rangle_b = \left\langle \int d^3p\ \hat{g} Q_{\rm EM}\right\rangle. \tag{89}$$

The definition of the passive cyclotron current drive efficiency (70) given in the previous section remains the same, however, the computation of the averaged toroidal current density from the radiation intensity now includes the poloidal averaging, as given by,

$$\langle j_t\rangle_S = \left\langle \int dK\ \eta_\omega^{(M)}(\theta,\vartheta) I_\omega^{(M)}(\theta,\Phi,\vartheta)\right\rangle. \tag{90}$$

For the purpose of the present research it is convenient to introduce a simplified model for the efficiency that allows us to study the effect of particle trapping on cyclotron current drive. Assuming that the source $Q_{\rm EM}$ is independent of the poloidal angle, one obtains for the current the following expression:

$$\langle j_t\rangle_S = \int dK \langle \eta_\omega^{(M)}\rangle_{\rm cyl}(\theta) I_\omega^{(M)}(\theta,\Phi), \tag{91}$$

where

$$\langle \eta_\omega^{(M)}\rangle_{\rm cyl} = \frac{4\pi^2 e^2}{m^2 c^3}\frac{\mu}{I_{\rm RJ}}\int dP\ A\hat{L}_n\langle \hat{g}\rangle. \tag{92}$$

This model is self-consistent with a straight-cylinder model of a tokamak in which the toroidal nonuniformity is neglected for the radiation problem, so that particle trapping is the only toroidal effect being included in this model.

VII. NUMERICAL METHOD AND RESULTS OF KINETIC MODELING

In the present paper a tokamak with concentric circular magnetic surfaces was chosen for the numerical evaluation of the Green's function $\hat{g}$. The integral part of the electromagnet interaction operator was omitted in the calculation because it plays the role of a correction term, the order of $1/\mu$ compared to the Fokker–Planck part of this operator.

The solution was obtained using the direct variational method for minimizing the functional (88).

The generalized Spitzer–Härm function $\hat{g}$ was presented as an expansion over the test functions,

$$\hat{g} = \sum_{i=0}^{N_p}\sum_{j=1}^{N_\chi}\sum_{l=0}^{2} \xi_{i,j,l} g_{i,j,l}, \tag{93}$$

where $g_{i,j,l}$ are the unknown coefficients of this expansion. The test functions are given as

$$\xi_{i,j,l}(p,\chi) = \Theta(p_0 - p_i)\Theta(p_i + \Delta p - p_0)(p_0 - p_i)^l P_j(\chi_0). \tag{94}$$

Here Θ is the Heaviside step function, $p_i = i\cdot\Delta p$, and Δp is a step in the momentum module.

The functions $P_j(\chi)$ are the odd eigenfunctions of the equation

$$\frac{1}{\sin \chi_0 K_1(\epsilon_t,\chi_0)} \frac{\partial}{\partial \chi_0} \sin \chi_0 K_2(\epsilon_t,\chi_0) \frac{\partial}{\partial \chi_0} P_j(\chi_0) + \lambda_j P_j(\chi_0) = 0, \quad (95)$$

which satisfy to the boundary conditions

$$\left.\frac{\partial P_j(\chi_0)}{\partial \chi_0}\right|_{\chi_0=0} = \left.\frac{\partial P_j(\chi_0)}{\partial \chi_0}\right|_{\chi_0=\pi} = 0, \quad (96)$$

$$P_j(\chi_b) = P_j(\pi - \chi_b) = 0, \quad (97)$$

in the domain $0 < \chi_0 < \chi_b$, $\pi - \chi_b < \chi_0 < \pi$, and equal to zero in the trapped region $\chi_b < \chi_0 < \pi - \chi_b$ (see e.g., Ref. 22). The functions K_1 and K_2 are defined as

$$K_1(\epsilon_t,\chi_0) = (1+\epsilon_t) \int_{-\pi}^{\pi} d\vartheta \, \frac{\cos \chi_0}{\cos \chi}, \quad (98)$$

$$K_2(\epsilon_t,\chi_0) = \int_{-\pi}^{\pi} d\vartheta (1+\epsilon_t \cos \vartheta) \frac{\cos \chi}{\cos \chi_0}, \quad (99)$$

with $\epsilon_t = r/R_0$.

The choice of the angular functions $P_j(\chi_0)$ is convenient for the description of the Coulomb collision operator because they are the eigenfunctions of the pitch-angle scattering operator.[23] In the particular case of a uniform magnetic field they correspond to the normalized Legendre polynomials.

The minimum condition of the functional (88) and the additional constraints on the continuity of the $\hat{g}$ function and its derivative over the momentum module,

$$C_{ij}^{(1)}(g_{i,j,l}, g_{i+1,j,l}) \equiv g_{i,j,0} + g_{i,j,1} \, \Delta p + g_{i,j,2} \, \Delta p^2 - g_{i+1,j,0} = 0, \quad (100)$$

$$C_{ij}^{(2)}(g_{i,j,l}, g_{i+1,j,l}) \equiv g_{i,j,1} + 2 g_{i,j,2} \, \Delta p - g_{i+1,j,1} = 0, \quad (101)$$

give the linear algebraic equation system for the coefficients $g_{i,j,l}$.

It should be mentioned here that the direct variational method of obtaining the generalized Spitzer–Härm distribution proposed in Ref. 15 is modified here, by using the quadratic splines as the components of the test functions over the momentum module variable. In Ref. 15 the solution was obtained in the form of a polynomial expansion, which fails at the high-energy region of the electron phase space. But this region is of main importance for the problem considered in the present paper.

The dependence of the generalized Spitzer–Härm function $\hat{g}$ on the momentum module is presented in Figs. 2 and 3 for two different values of the electron pitch angle χ_0 in the case of a uniform magnetic field ($\epsilon_t = 0$). One can see that the effect of collisions due to radiation (mainly the radiation friction force) is strong for low-β plasmas with high temperature ($\mu = 6$ corresponds to the temperature close to 80 keV), where

$$\beta_e = \frac{8 \pi n_e T_e}{B_0^2} \quad (102)$$

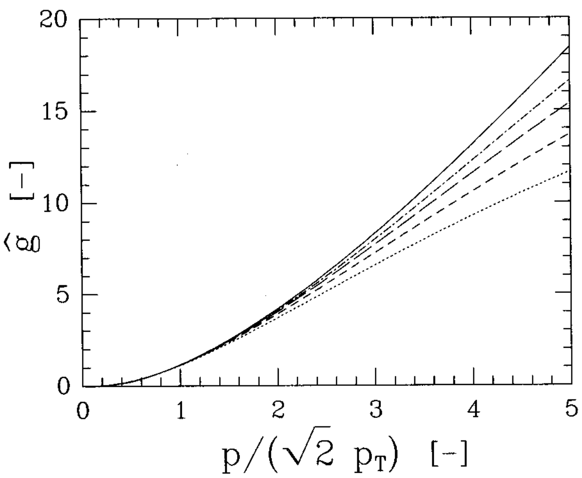

FIG. 2. The $\hat{g}$ function versus the dimensionless electron momentum. Here the pitch angle $\chi = \pi/8$; the inverse temperature $\mu = 6.0$; $\beta_e = \infty$ (solid line); $\beta_e = 0.08$ (dash–dotted line); $\beta_e = 0.04$ (long-dashed line); $\beta_e = 0.02$ (short-dashed line); and $\beta_e = 0.01$ (dotted line).

is the electron pressure. At a fixed plasma temperature the ratio between the radiation collision operator and the Coulomb collision operator is inversely proportional to β_e. The depletion of the $\hat{g}$ function in the high-energy region compared to the pure Coulomb relaxation case ($\beta_e = \infty$) is stronger for pitch angles close to $\pi/2$. This follows from the anisotropy of the radiation collision operator.

The intensity related efficiency η (92) was numerically calculated for the whole range of frequencies and radiation pitch angles with the help of the previously obtained Spitzer–Härm function without the use of the asymptotical formulas for Bessel functions. The behavior of this quantity as a function of the radiation pitch angle is shown in Figs. 4

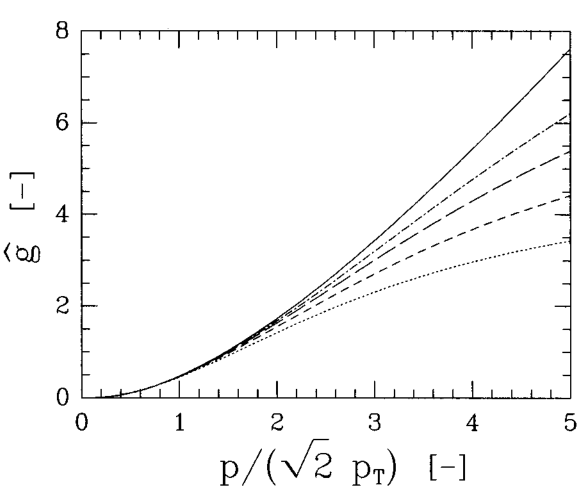

FIG. 3. The $\hat{g}$ function versus the dimensionless electron momentum. Here the pitch angle $\chi = 3\pi/8$; the inverse temperature $\mu = 6.0$; $\beta_e = \infty$ (solid line); $\beta_e = 0.08$ (dash–dotted line); $\beta_e = 0.04$ (long-dashed line); $\beta_e = 0.02$ (short-dashed line); and $\beta_e = 0.01$ (dotted line).

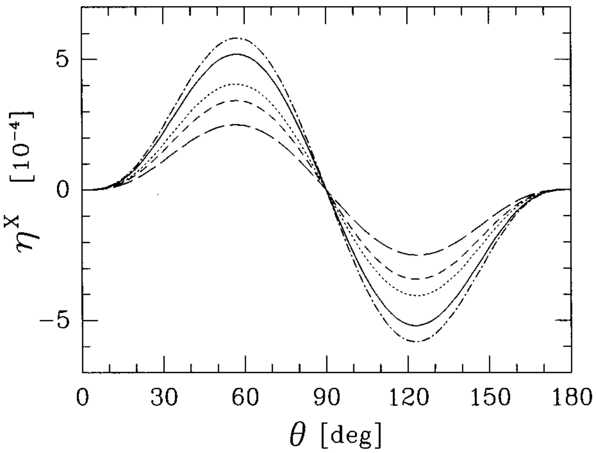

FIG. 4. Passive current drive efficiencies for the X mode as given by different calculation models versus wave pitch angle Θ. Here μ=6.0; $\overline{\omega}$=7.5; ϵ_t=0.0; η for $\beta_e=\infty$ (dash–dotted line); η for β_e=0.08 (solid line); η for β_e=0.02 (dotted line); η_S (short-dashed line); and η_F (long-dashed line).

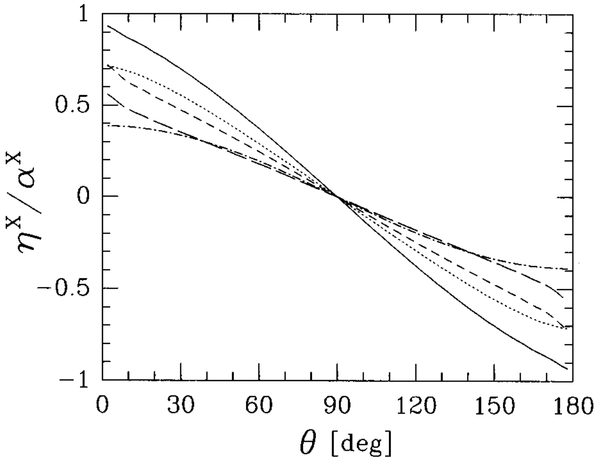

FIG. 6. Ratio between absorption coefficient and current drive efficiency $G=\alpha/\eta$ for the X mode as given by different calculation models versus wave pitch angle Θ. Here μ=6.0; $\overline{\omega}$=7.5; ϵ_t=0.0; η/α for β_e=0.08 (solid line); η/α for β_e=0.02 (dotted line); η_S/α (short-dashed line); η_F/α (long-dashed line); and η_{FA}/α (dashed-dotted line).

and 5. Everywhere in the plots the dimensionless wave frequency $\overline{\omega}=\omega/\omega_{ce}$ is used as a parameter.

The quantity η_S is calculated without both the integral part of the Coulomb collision operator and the radiation friction force. The numerical computation of the quantity η_F is based on the asymptotical formula for the Spitzer–Härm function $g=ev_\parallel/\nu_{\text{eff}}$ used in Ref. 2, instead of the function $\hat{g}$ derived in this paper. Here ν_{eff} is the effective collision frequency introduced by Fisch.[8]

As one can see from Figs. 4 and 5 the values of the passive cyclotron current drive efficiency given by the exact model of relaxation are noticeably higher than those given by the asymptotical model. This is due to the increased role of the response of the background electrons to the perturbation of the electron distribution function in the high-energy region (the integral part of the Coulomb collision operator) at relativistic plasma temperatures. This effect is completely neglected in the asymptotical model. The effect of collisions due to radiation is obviously important for low-pressure plasmas, e.g., for β_e=0.02 the positive effect of the integral part is almost canceled by the negative effect of the radiation friction force. However, at β_e=0.08 it is already very small.

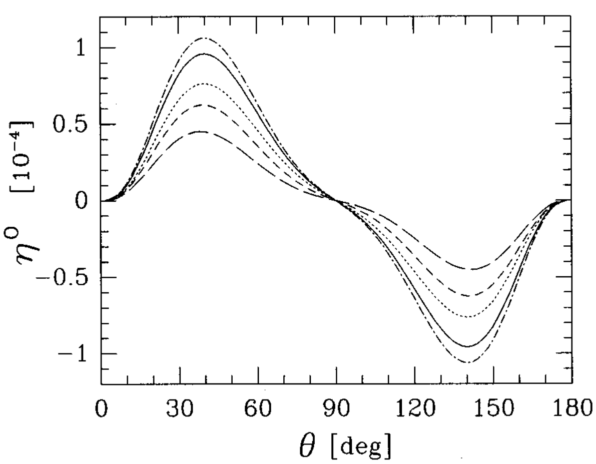

FIG. 5. Passive current drive efficiencies for the O mode as given by different calculation models versus wave pitch angle Θ. Here μ=6.0; $\overline{\omega}$=7.5; ϵ_t=0.0; η for $\beta_e=\infty$ (dash–dotted line); η for β_e=0.08 (solid line); η for β_e=0.02 (dotted line); η_S (short-dashed line); and η_F (long-dashed line).

An interesting fact is, that even without the use of the integral part of the electron–electron Coulomb collision operator (background response), one obtains somewhat higher efficiencies (η_S) as compared to those calculated with the help of the asymptotical formula η_F. This is mainly due to the increased role of diffusion over energy for a relativistic plasma (see e.g., Refs. 24 and 4), as well as due to the decrease of the pitch-angle scattering coefficient due to thermal effects. These thermal effects were not taken into account in the early reference by Fisch[9] used in Ref. 2. This effect is obvious from the formulas (35) of Ref. 16, in which the ratio of the thermal velocity to the test particle velocity cannot be considered to be small for relativistic plasmas because both the nominator and the denominator approach the speed of light.

The passive current drive efficiencies for the X mode are roughly five times as high as those for the O mode, finally resulting in a much higher contribution of the X mode to the passively generated plasma current. This follows from the well-known fact of the X mode being the main contributor to the cyclotron power loss.

For a more detailed comparison with the results of Ref. 2, the ratio of the efficiency η and the absorption coefficient α is plotted in Fig. 6. This ratio—G—was introduced in Ref. 2 and then used in Refs. 3–5. The results were again calculated for the different models of relaxation. One should note that the agreement between the approximate formula for

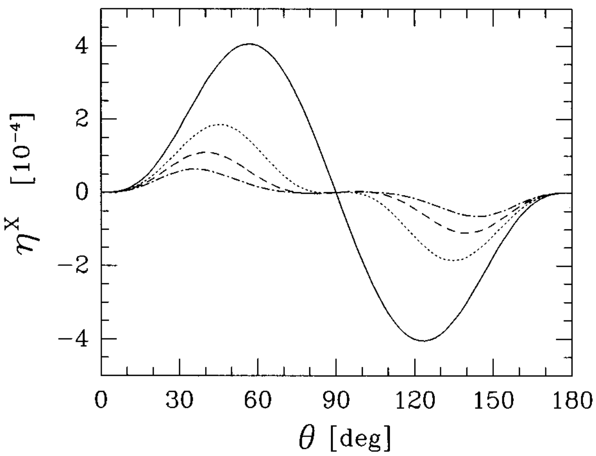

FIG. 7. Influence of toroidicity on the passive current drive efficiency η for the X mode. Here μ=6.0; $\overline{\omega}$=7.5; β_e=0.02; ϵ_t=0.0 (solid line); ϵ_t=0.1 (dotted line); ϵ_t=0.2 (dashed line); and ϵ_t=0.3 (dash–dotted line).

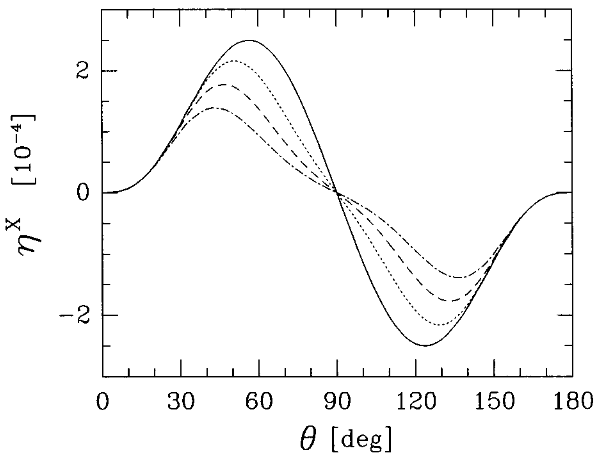

FIG. 8. Influence of toroidicity on the passive current drive efficiency η for the X mode. Here the asymptotical approach for the Coulomb collision operator was used. Here μ=6.0; $\overline{\omega}$=7.5; ϵ_t=0.0 (solid line); ϵ_t=0.1 (dotted line); ϵ_t=0.2 (dashed line); and ϵ_t=0.3 (dash–dotted line).

$G=\eta_{FA}/\alpha$ given in Ref. 2 and its equivalent using Fisch's asymptotical relaxation model η_F/α is very good in the range of the radiation pitch angle $\Theta\approx 40°$ of main interest for that study. In addition to this, it has to be noted that the behavior of this ratio η/α is very similar for the X and the O mode.

In Fig. 7 the influence of the toroidicity on the current drive efficiency is demonstrated. One may notice that this influence is not very dramatic. However, even for small values of the inverse aspect ratio ϵ_t it is not negligible. It results both from the rapid decrease of the additional current produced by the response of background electrons and from the increase of the momentum loss cone in the phase space corresponding to the region of trapped particles (see, e.g., Refs. 22, 24, and 25).

In order to clarify the role of the loss cone, the calculations were performed with the asymptotic Fisch formula (see above) in which the loss cone region was excluded from the momentum space integration. The results for this quantity are presented in Fig. 8. One can easily see that this effect alone does not produce such a dramatic decrease.

Finally, in Fig. 9 the behavior of the efficiency in a D–^{3}He plasma with different concentrations of ^{3}He is shown. In changing the ^{3}He concentration, the electron pressure is kept constant. The study of the dependence of the efficiency on the ^{3}He concentration shows a somewhat stronger dependence on the effective charge number than the $1/(5+Z)$ law. This comes from the fact that the behavior of the integral part of the collision operator, which plays a positive role for the current efficiency, has a somewhat different scaling with the charge number. The decrease of the current with the charge number, however, is not dramatic. That leaves the mechanism under consideration to be of interest for a future D–^{3}He reactor.

VIII. CONCLUSIONS

The main feature of the present study of the passive cyclotron current drive efficiency is the account of the following effects: Response of the background electrons; the effect of ''collisions'' due to radiation (mainly the radiation drag); and the effect of toroidicity.

Most of these effects (except the diffusion of particles in the velocity space due to ''thermal'' radiation) were accounted for in earlier studies of other methods of current drive. However, they were not considered in the case of passive cyclotron current drive. The computer modeling of the passive cyclotron current drive efficiency is performed in the present paper without asymptotical approximations for the Spitzer–Härm function and the quasilinear operator. It shows that all three effects have an important influence on the passive cyclotron current drive efficiency. So, all three effects should be taken into account for a self-consistent modeling of passive current generation. The results of such a

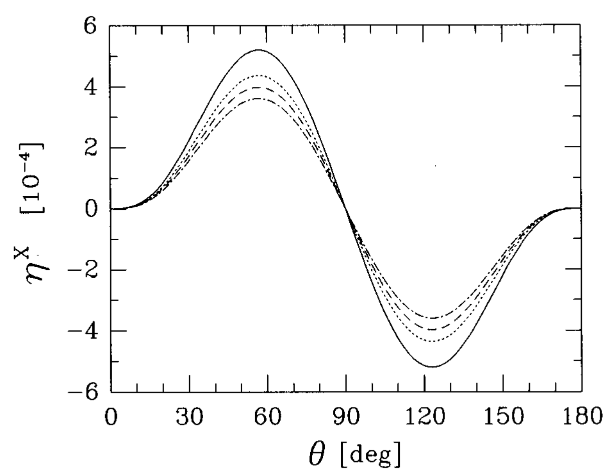

FIG. 9. Influence of ^{3}He concentration on the passive current drive efficiency. Here μ=6.0; $\overline{\omega}$=7.5; β_e=0.08; ϵ_t=0.0; $n_{^3He}/n_D$ = 0.0 (solid line); $n_{^3He}/n_D$ = 0.25 (dotted line); $n_{^3He}/n_D$ = 0.5 (dashed line); and $n_{^3He}/n_D$ = 1.0 (dash–dotted line).

modeling[6,7] show that the absolute value as well as the current density profile are strongly influenced by these effects.

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